The University of Hohenheim
Chair of Banking and Financial Services

Portfolio Management

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Part I:

Basics of Portfolio Selection Theory

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Agenda

- **Portfolio Management:** planning, management and control of capital investments
- Focus: Portfolio asset allocation

  ➡ **Portfolio Theory**

- Harry M. **Markowitz** (1952):

  Normative model of **Portfolio Selection** for stock portfolios

  Derivation of an optimal allocation of wealth to various securities

  Explanation of investor behavior

  Alternatives: Portfolio Selection with and without a risk-free asset

- William F. Sharpe (1964):

  Capital market model based on Portfolio Selection Theory

  Derivation of risk premiums and security prices

  ➡ **Capital Asset Pricing Model**

Common characteristics of these approaches

- Assets are assessed based on return and risk
- Return is described by the expected rate of return μ on the security/portfolio
- Risk is described by the standard deviation σ of the expected rate of return on the security/portfolio

→ **μ-σ-principle**

Questions:

- Is it possible to derive a general rule for investment behavior which only takes into account the μ and σ characteristics of the portfolio by maximizing individual utility?
- Why is the rate of return relevant for Portfolio Selection and not the chart-development of a stock?

Calculation of the rate of return

$$\text{Rate of return (\%)} = \frac{\text{Profit}}{\text{Capital invested}} \times 100$$

Discrete rate of return for one period:

$$r_{d,t} = \frac{C_t - C_{t-1}}{C_{t-1}} + \frac{D_t}{C_{t-1}} = \frac{C_t + D_t}{C_{t-1}} - 1$$

Return on capital + Dividend yield = Total yield

$r_{d,t}$ = discrete rate of return in period t

C_t = capital at the end of the period

C_{t-1} = capital at the beginning of the period

D_t = matured dividends in period t

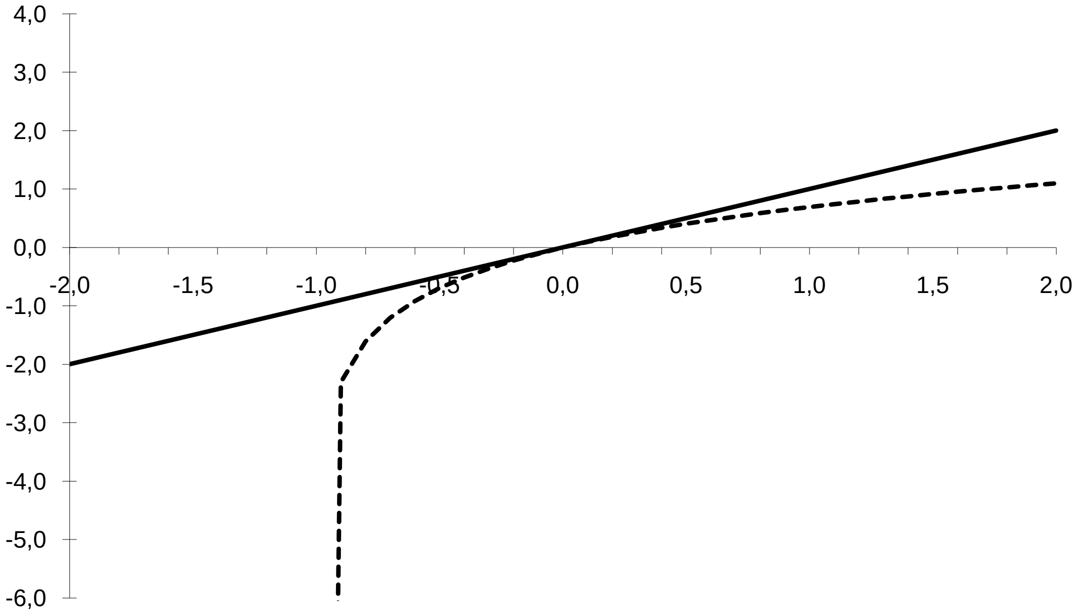
Continuous rate of return for one period:

$$r_{s,t} = \ln\left(\frac{C_t + D_t}{C_{t-1}}\right) = \ln\left(C_t + D_t\right) - \ln C_{t-1}$$

- Discrete and continuous rates of return can be mutually converted:

$$r_{s,t} = \ln(1 + r_{d,t}) \quad \text{and} \quad r_{d,t} = e^{r_{s,t}} - 1$$

- Discrete and continuous rate of returns on the interval [-2,2]



- $r_s < r_d$
- The difference between discrete and continuous rate of returns increases in magnitude
- For very small values of r_s and r_d: $r_s \approx r_d$

Advantages of using continuous rates of return

- Capital is compounded continuously (stock price formation as a continuous process)
- Discretionary small periods of interest calculation, discretionary large number of observation periods
- Simple calculation of rates of return for multiple periods (addition of single period returns) and the average return (arithmetic mean)
- Continuous rates of return imply that stock prices cannot become negative
- Empirically: continuous rates of return are rather normally distributed than discrete rates of return

→ In capital market research logarithmic returns are commonly used.

Problems in calculating rates of return

- Dividend payments in period T require an assumption about reinvestment in this period
- If corporate actions (increase in capital, decrease of capital, stock splits etc.) take place during the observation period T stock price adjustments are required.

1) Accounting for the value of rights issue:

$$RI = \frac{C_{before,t} - IP}{SR + 1} \quad \text{respectively} \quad RI = \frac{C_{after,t} - IP}{SR}$$

2) Retrograde revision of stock prices

Reinvestment of rights issue in new stock without additional capital is assumed (Operation Blanche)

Stock prices before corporate actions are multiplied by an adjustment factor

$$AF = \frac{C_{after,t}}{C_{after,t} + D}$$

D = Dividends or other calculative deductions

Example:

Period	Date	Price	Corporate Action
5	01/02/2007	270	
4	04/01/2007	288 exDiv	Dividend payout 12
3	01/02/2006	320	
2	10/01/2005	354 exRI	Regular increase in capital in relation 4:1; Issuing price 170; new stock is fully entitled to dividends from 01/01/2005
1	07/01/2005	360	

Source: Steiner/Uhlir (2001), pp. 126 et seq.

Aggregation of returns

over a time horizon ΔT with t=1 to N periods

	Discrete rate of return	Continuous rate of return
Total rate of return in T	$r_{d,\Delta T} = \prod_{t=1}^{N} (1 + r_{d,t}) - 1$	$r_{s,\Delta T} = \sum_{t=1}^{N} r_{s,t}$
Average rate of return in T for t	$\overline{r_{d,\Delta T}} = \sqrt[N]{\prod_{t=1}^{N} (1 + r_{d,t})} - 1$	$\overline{r_{s,\Delta T}} = \frac{1}{N} \sum_{t=1}^{N} r_{s,t} = \frac{1}{N} [\ln K_T - \ln K_0]$

- geometric calculation
- arithmetic calculation

Evaluation of portfolio returns

x_i = proportion of security i in the portfolio

Discrete portfolio return	Continuous portfolio return
$r_{d,P} = \sum_{i=1}^{I} x_i \cdot r_{d,i}$	$r_{s,P} = \ln\left(\sum_{i=1}^{I} x_i \cdot e^{r_{s,i}}\right)$

- Continuous rate of return on different securities cannot be summarized to a single portfolio rate of return
- The continuous portfolio rate of return is not the weighted average of the continuous rates of return (the logarithm of a sum of values is different from a sum of logarithmic values)

For returns close to zero the difference between continuous and discrete rates of return is marginal

$$r_{s,P} \approx \sum_{i=1}^{I} x_i \cdot r_{s,i}$$

Expected value, standard deviation and variance

p_k = probability of state k

r_k = continuously compounded annual rate of return in state k

- Expectation value of the continuously compounded annual rate of return

$$E(r_s) = \mu_s = \sum_{k=1}^{K} p_k \cdot r_{s,k}$$

- Variance of the continuously compounded annual rate of return

$$Var(r_s) = \sigma_s^2 = \sum_{k=1}^{K} p_k (r_{s,k} - \mu_s)^2$$

- Standard deviation of the continuously compounded annual rate of return

$$\sigma_s = \sqrt{\sigma_s^2}$$

Relationship between rates of return of two securities i and j

Covariance and correlation describe the strength and direction of the relationship between rates of return of the two stocks

- Covariance between the continuously compounded annual rates of return i and j:

$$cov(r_{s,i}, r_{s,j}) = \sigma_{s,ij} = \sum_{k=1}^{K} p_k (r_{s,i,k} - \mu_{s,i,k})(r_{s,j,k} - \mu_{s,,k})$$

- Correlation between the continuously compounded annual rates of return i and j:

$$\rho_{s,ij} = \frac{\sigma_{s,ij}}{\sigma_{s,i}\sigma_{s,j}}$$

Advantage of the correlation coefficient over the covariance is its scaling on

$$-1 \leq \rho_{s,ij} \leq 1$$

Expected value of portfolio returns

- The expected value of portfolio returns can be computed in two different ways
  - via state-dependent portfolio returns

$$E(r_P) = \mu_P = \sum_{s=1}^{S} p_s \cdot r_{P,s} \qquad \text{with} \qquad r_{P,s} = \sum_{i=1}^{N} x_i \cdot r_{i,s}$$

  - via expected values of stock returns

$$E(r_P) = \mu_P = \sum_{i=1}^{N} x_i \cdot \mu_i$$

Variance of portfolio returns

- The variance of portfolio returns can be computed in three different ways:
  - via state-dependent portfolio returns

$$\mathrm{Var}(r_P) = \sigma_P^2 = \sum_{s=1}^{S} p_s (r_{P,s} - \mu_P)^2$$

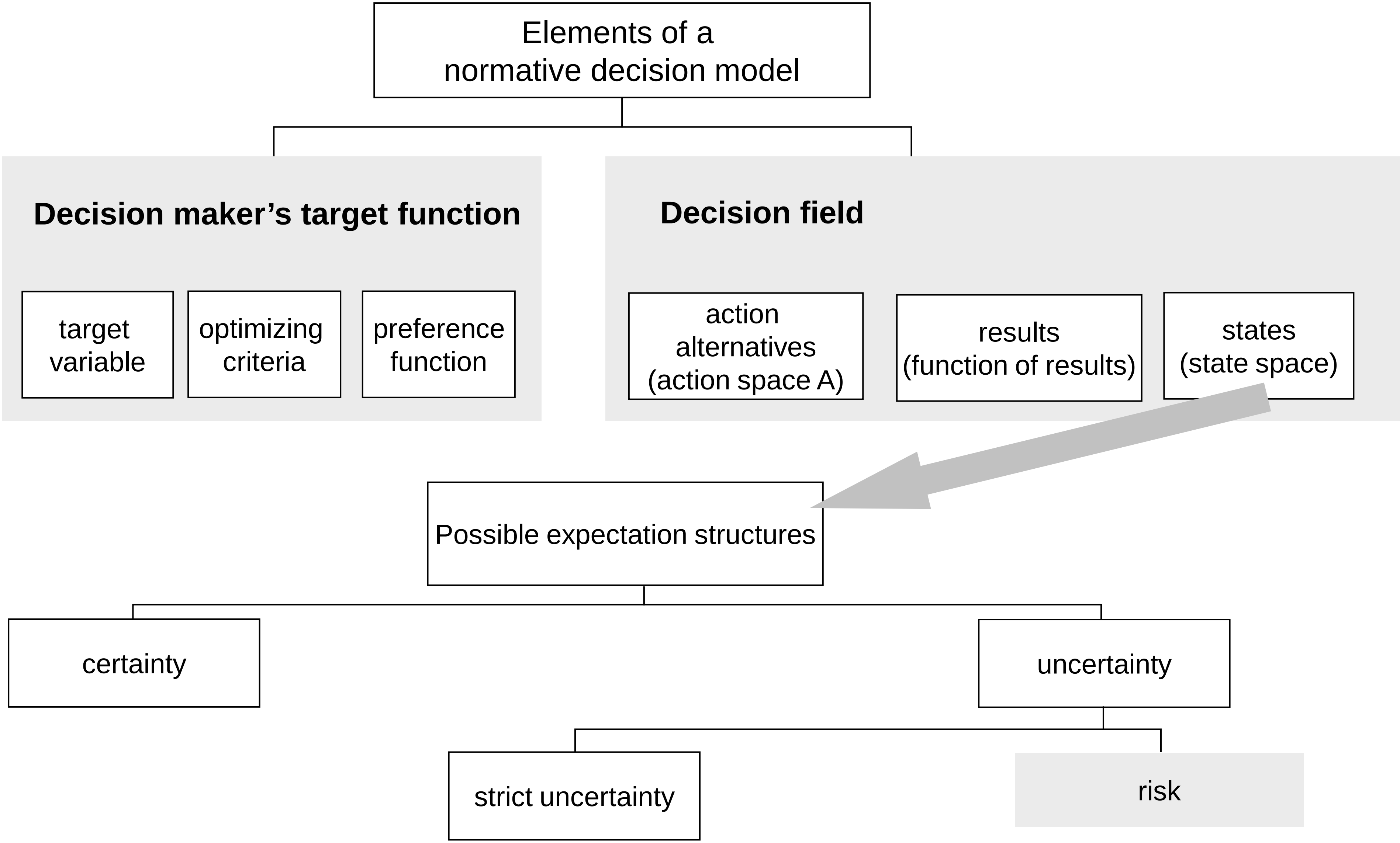
  - via the variance-/covariance-matrix of security returns

$$\mathrm{Var}(r_P) = \sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{i,j}$$

  - via the covariance of the security return with the portfolio return

$$\mathrm{Var}(r_P) = \sigma_P^2 = \mathrm{cov}(r_P, r_P) = \mathrm{cov}\left(\sum_{i=1}^{N} x_i r_i, \sum_{j=1}^{N} x_j r_j\right)$$

$$= \sum_{i=1}^{N} x_i \cdot \mathrm{cov}\left(r_i, \sum_{j=1}^{N} x_j r_j\right) = \sum_{i=1}^{N} x_i \cdot \mathrm{cov}(r_i, r_P)$$


Elements of a normative decision model
Decision maker's target function
target variable
optimizing criteria
preference function
Decision field
action alternatives (action space A)
results (function of results)
states (state space)
Possible expectation structures
certainty
uncertainty
strict uncertainty
risk

- target variable
- preference function: $\Phi(A)$
- optimization criterion: $\phi(a^*) = \max_{a_i \in A} \phi(a_i)$

- target variable, preference-function and optimization criterion determine a decision rule
- the formal description of a decision rule is the target-function

- specification of the preference function via a decision criterion
- classical decision criteria under risk:
  - μ-principle
  - μ-δ-principle
  - Bernoulli-principle

Bernoulli-Principle

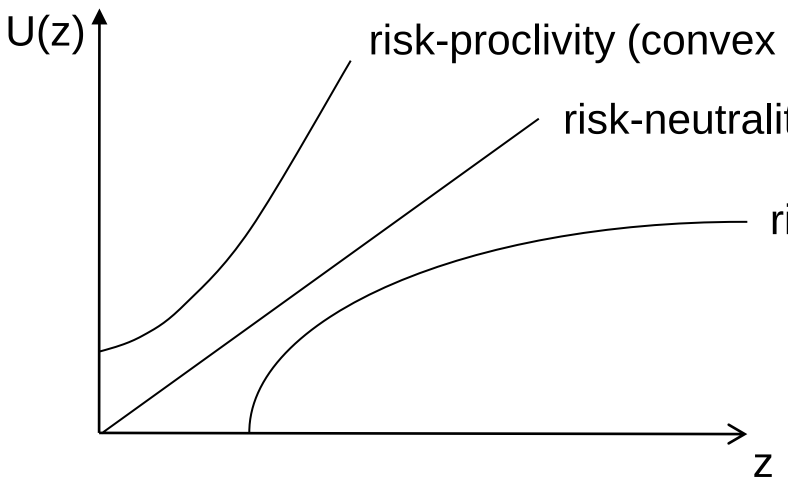
➢ General approach:

- by using a risk-utility function u every result is assigned to a real utility value $u(e_{is})$
- the expectation value of the utility values (not of the results) is determined for each alternative
- the alternative with the highest expected utility value is selected

	s_1	s_2	...	s_s
	p_1	P_2	...	p_s
a_1	e_{11}	e_{12}	...	e_{1s}
a_2	e_{21}	e_{22}	...	e_{2s}
.	.	.		.
a_a	e_{a1}	e_{a2}	...	e_{as}

$$\phi(a_i) = E[u(e_{is})] = \sum_{s=1}^{S} p_s u(e_{is})$$

Detailed analysis of the risk-utility function



- diminishing marginal utility indicates risk-aversion, constant marginal utility indicates risk-neutrality and increasing marginal utility indicates risk-proclivity
- a positive linear transformation of the utility-function does not alter the ranking order of the alternatives' utility expectation values: $u^*(e_{is}) = c \cdot u(e_{is}) + d, c > 0$

➡ location and slope of the risk-utility function are not of interest

➡ for the assessment of the risk-utility function the curvature is the relevant criterion

μ-σ-Principle

Preference function: $\Phi(a_i) = \Phi(\mu_i, \sigma_i)$

Example: $\Phi(a_i) = \mu_i + \alpha\sigma_I$ with

$\alpha < 0 \leftrightarrow$ risk-aversion

$\alpha = 0 \leftrightarrow$ risk-neutrality

$\alpha > 0 \leftrightarrow$ risk-prolictivity

Advantages:

- two-parametrical decision calculus
- knowledge of the utility function or complete configuration of preferences obsolete (?)

Common assumption on capital markets: risk-aversion

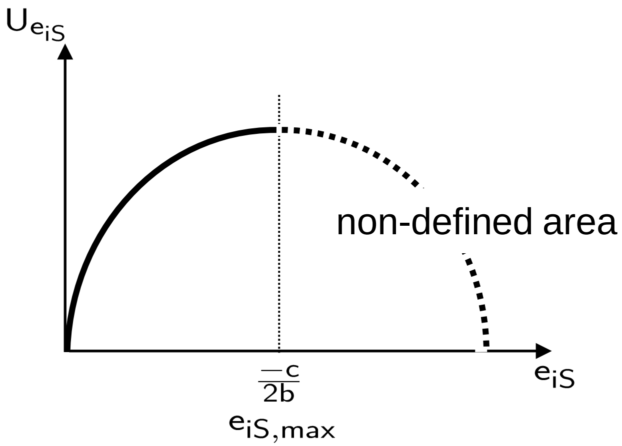
Accordance of the μ-δ-principle with the Bernoulli-principle

Accordance of those principles is only apparent in the presence of restricting assumptions regarding the utility function of the decision maker and/or the distribution function of the target variable:

- the risk-utility function is squared:

$$u(e_{is}) = be_{is}^2 + ce_{is}$$

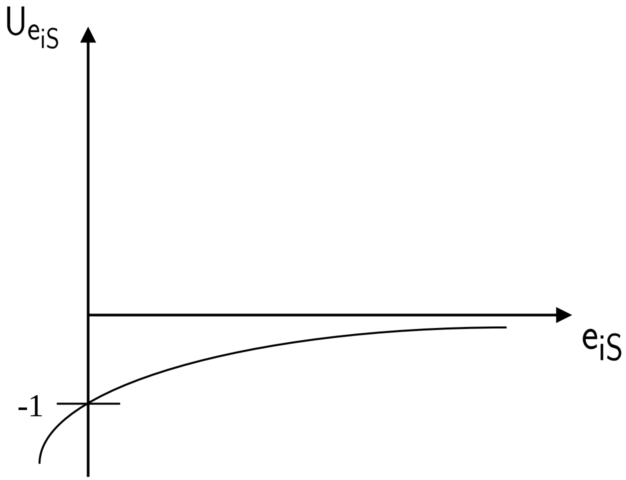
with b<0 and c>0



- the risk-utility function is exponential **and** the target variable is normally distributed:

$$u(e_{is}) = -e^{-ae_{is}}$$

with a>0



Expected utility value and μ-δ-principle

Only if the above specified assumptions hold and under consideration of the Bernoulli-principle it is possible to revert to the more simple μ-δ-principle.

The preference functions are as follows:

- $\phi(A) = E[u(e)] = b[\sigma^2 + \mu^2] + c\mu$

  for the squared risk-utility function

- $\phi(A) = E[u(e)] = \mu - \frac{a}{2} \cdot \sigma^2$

  for the exponential risk-utility function

Aim of Portfolio Selection:

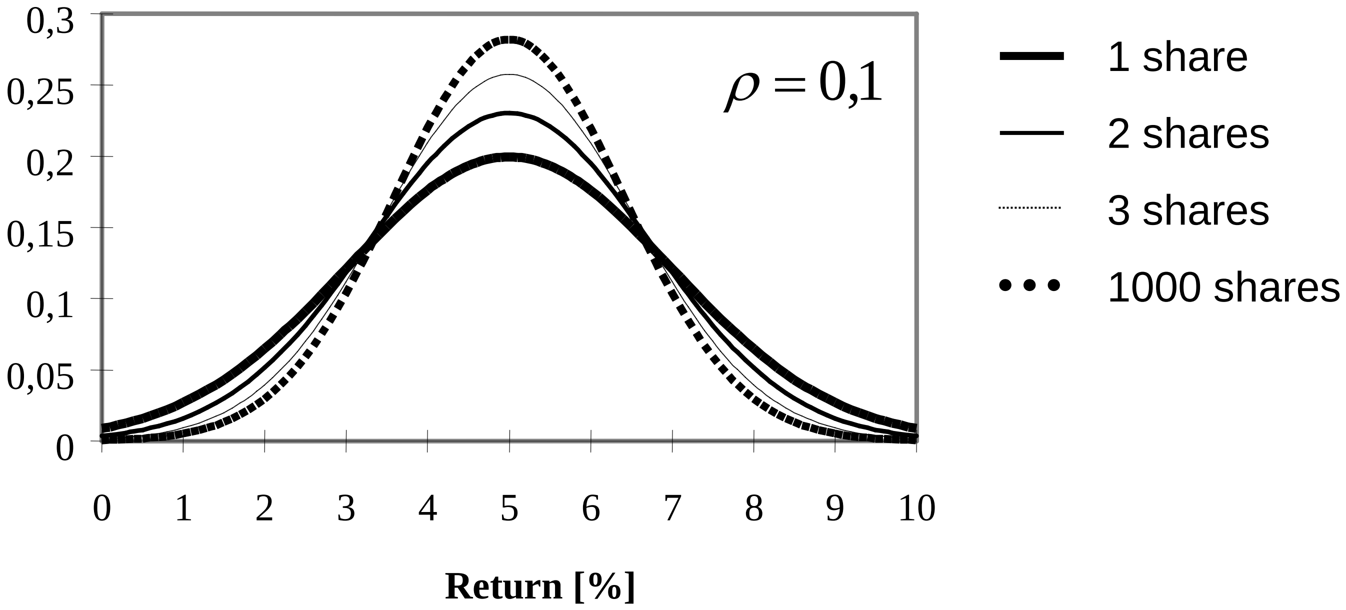
Efficient allocation of given funds to multiple securities

- The variation of portfolio returns depends on both, the variation in return of individual securities and the return correlation between these individual securities
- If correlations between these securities are incomplete, diversification leads to risk reduction
- Risk reduction through diversification can be achieved by simply allocating funds to various assets (naive diversification)
- Following an optimization calculus, the proportion of funds allocated to individual assets are so chosen as to generate an optimal risk-chance-position according to the μ-δ-principle (portfolio selection)

Naive Diversification

Assumption: Returns are identically normally distributed and stochastically independent

If funds are equally distributed between various assets (naive diversification), the probability distribution of portfolio return is as follows:



Diversification is possible for $\rho < 1$

With $\rho = 0$ risk can be completely eliminated with an infinite number of assets

With $\rho > 0$ an element of risk remains which cannot be reduced further by adding more assets

Naive Diversification for uncorrelated returns

It is $x_i = \frac{1}{N}$ and $\sigma_{ij} = 0$

Firstly, we assume that the returns of all stocks i ≠ j are independent from each other.

Hence, the variance is determined as follows:

$$\sigma_P^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij} = \sum_{i=1}^{N} x_i^2 \sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ i \neq j}}^{N} x_i x_j \sigma_{ij} = \sum_{i=1}^{N} x_i^2 \sigma_i^2$$

$$= \sum_{i=1}^{N} \left(\frac{1}{N}\right)^2 \sigma_i^2 = \frac{1}{N}\left[\sum_{i=1}^{N} \frac{\sigma_i^2}{N}\right] = \frac{1}{N} \overline{\sigma}_i^2$$

If an increasing number of stocks is added to the portfolio the portfolio variance becomes:

$$\lim_{N \to \infty} \sigma_P^2 = \lim_{N \to \infty} \left(\frac{1}{N} \cdot \overline{\sigma}_i^2\right) = 0$$

Naive Diversification for positively correlated returns

Usually, stock returns are positively correlated, i.e.: $\sigma_{ij} > 0$

Taking correlation into account the variance of the portfolio is computed as follows:

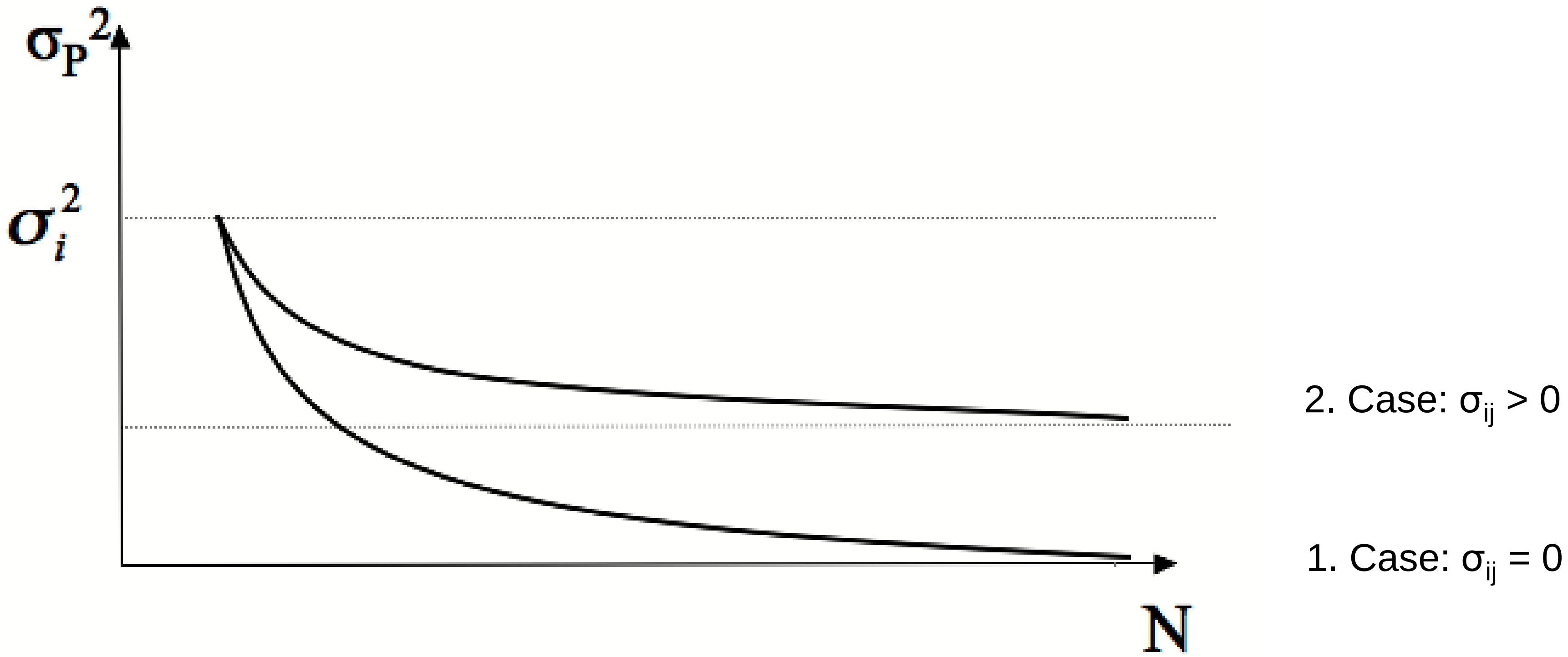
$$\sigma_P^2 = \sum_{i=1}^{N}\left(\frac{1}{N}\right)^2 \sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N}\left(\frac{1}{N}\right)\left(\frac{1}{N}\right)\sigma_{ij} = \frac{1}{N}\left(\sum_{i=1}^{N}\frac{\sigma_i^2}{N}\right) + \frac{N-1}{N}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N}\frac{\sigma_{ij}}{N\cdot(N-1)}$$

$$= \frac{1}{N}\overline{\sigma}_i^2 + \frac{N-1}{N}\overline{\sigma}_{ij} = \frac{1}{N}\left(\overline{\sigma}_i^2 - \overline{\sigma}_{ij}\right) + \overline{\sigma}_{ij}$$

The portfolio variance for N → ∞ is:

$$\lim_{N\to\infty}\sigma_P^2 = \lim_{N\to\infty}\left[\frac{1}{N}\left(\overline{\sigma}_i^2 - \overline{\sigma}_{ij}\right) + \overline{\sigma}_{ij}\right] = \overline{\sigma}_{ij}$$

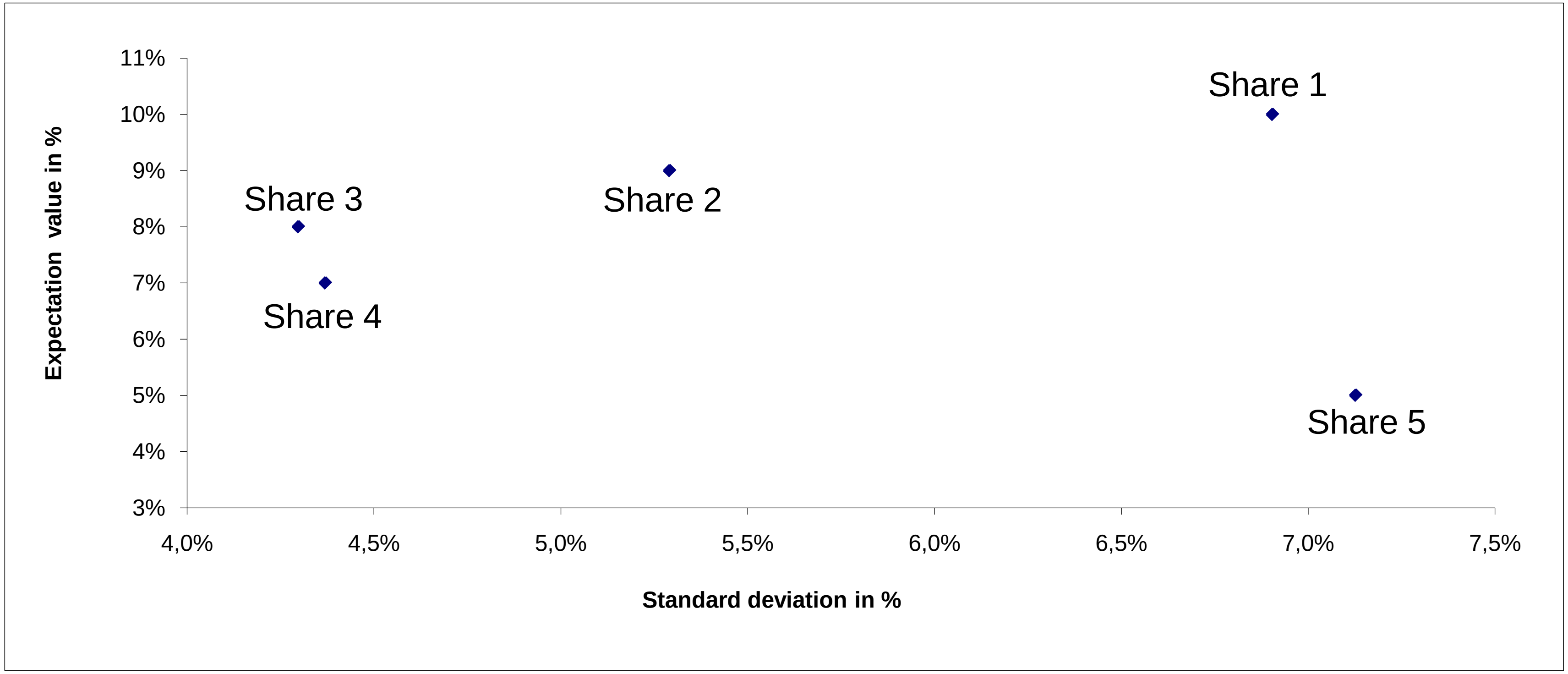
Comparison of the effects of naive diversification depending on the correlation of returns



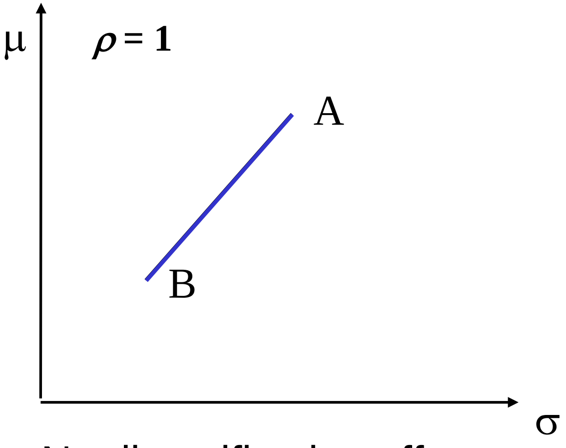
Assumptions

1. The investor maximizes his expected utility (Bernoulli-utility)
2. The marginal utility is always positive
3. The investor evaluates securities and portfolios based on expected value and standard deviation (respectively variance) of returns
4. The investor is risk-averse
5. The investor's investment-horizon is one period
6. The number of securities and the probability distributions of future returns are exogenous and known to the investor
7. No transaction costs, no taxes, no indivisibility problems, perfect competition (→ no arbitrage opportunities)

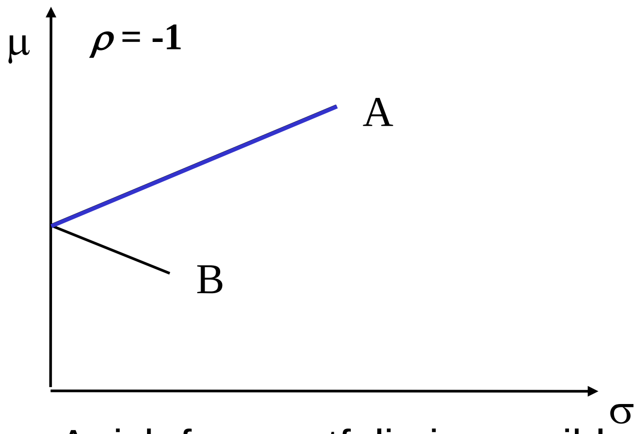
Efficient and optimal portfolios



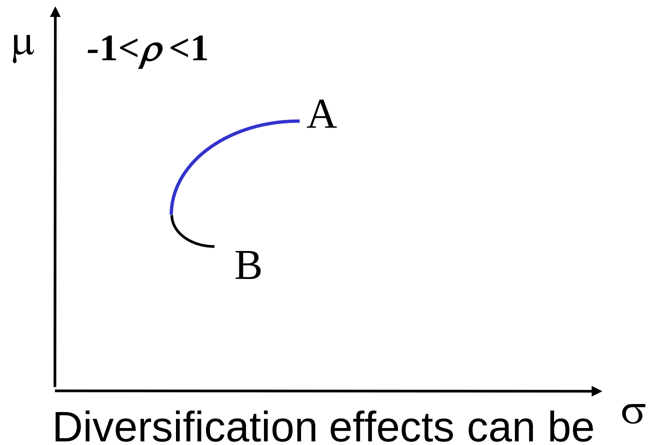
- With continuous variation of the portfolio-allocation all the (μ-σ)-combinations can be achieved that are part of a connection line derived by connecting the (μ-σ)-specifications between the individual stocks.
- The set of all possible (μ-σ)-combinations is called ***feasible set***.



No diversification effects



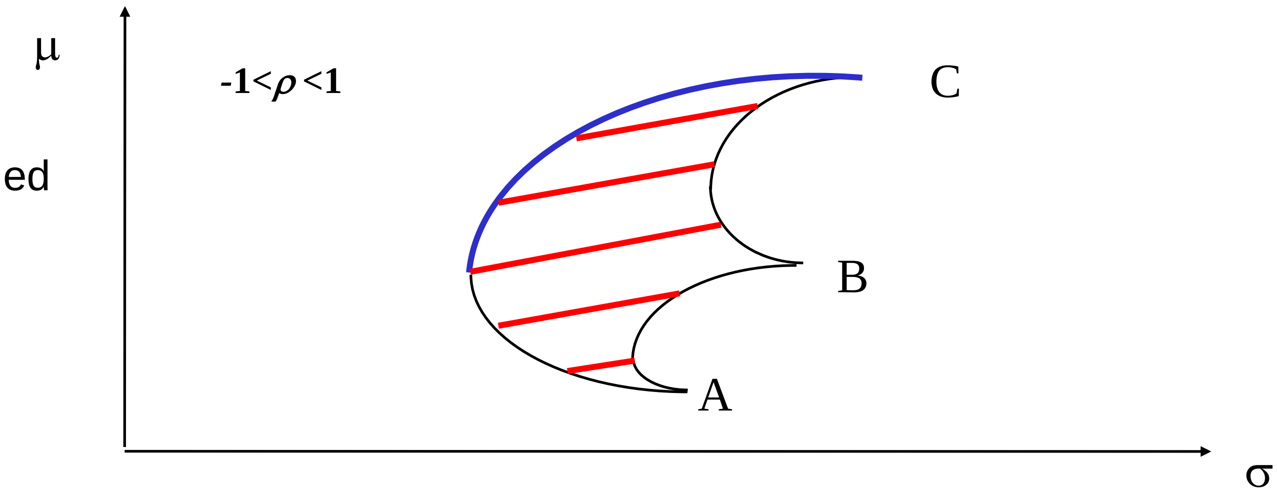
A risk-free portfolio is possible



Diversification effects can be realized

The set of all efficient portfolios is called

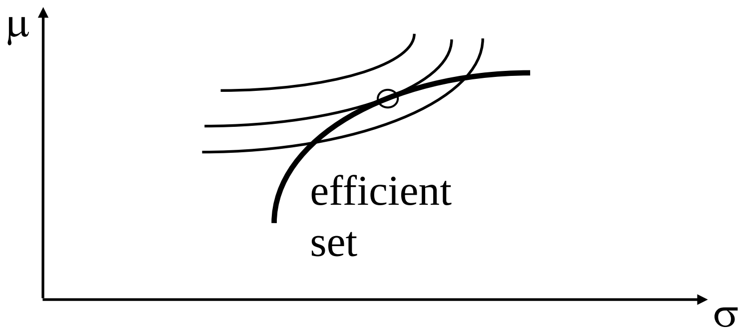
efficient set / efficient frontier.



Extreme cases are given when ρ = 1 and ρ = -1

Optimal Portfolio

- The selection of an optimal portfolio from the set of all efficient portfolios is only possible if investor's utility- or preference-function is taken into account
- The optimal portfolio can be different for each investor depending on its individual preference-function



- The following optimization problem must be solved: for a given expected value of the portfolio's return μ_p the combination of shares $\{x_1,..., x_N\}$ has to be found which minimizes the portfolio's variance
- The following side conditions must be considered

$$\sum_{i=1}^{N} x_i = 1 \quad \text{(budget constraint) and} \quad \bar{\mu}_P = \sum_{i=1}^{N} x_i \mu_i$$

- The Lagrange approach is used to solve the problem:

$$L = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij} + \lambda_1\left(\sum_{i=1}^{N} x_i \mu_i - \bar{\mu}_P\right) + \lambda_2\left(\sum_{i=1}^{N} x_i - 1\right)$$

Example 1

The investor sets up a portfolio with only two shares, share 1 and share 2. The expected returns for share 1 and share 2 are $E(r_1) = \mu_1 = 0{,}2$ and $E(r_2) = \mu_2 = 0{,}3$. The variances for these shares are $var(r_1) = \sigma_1^2 = 0{,}04$, $var(r_2) = \sigma_2^2 = 0{,}08$. The covariance between share 1 and share 2 is $cov(r_1,r_2) = \sigma_{12} = 0{,}02$.

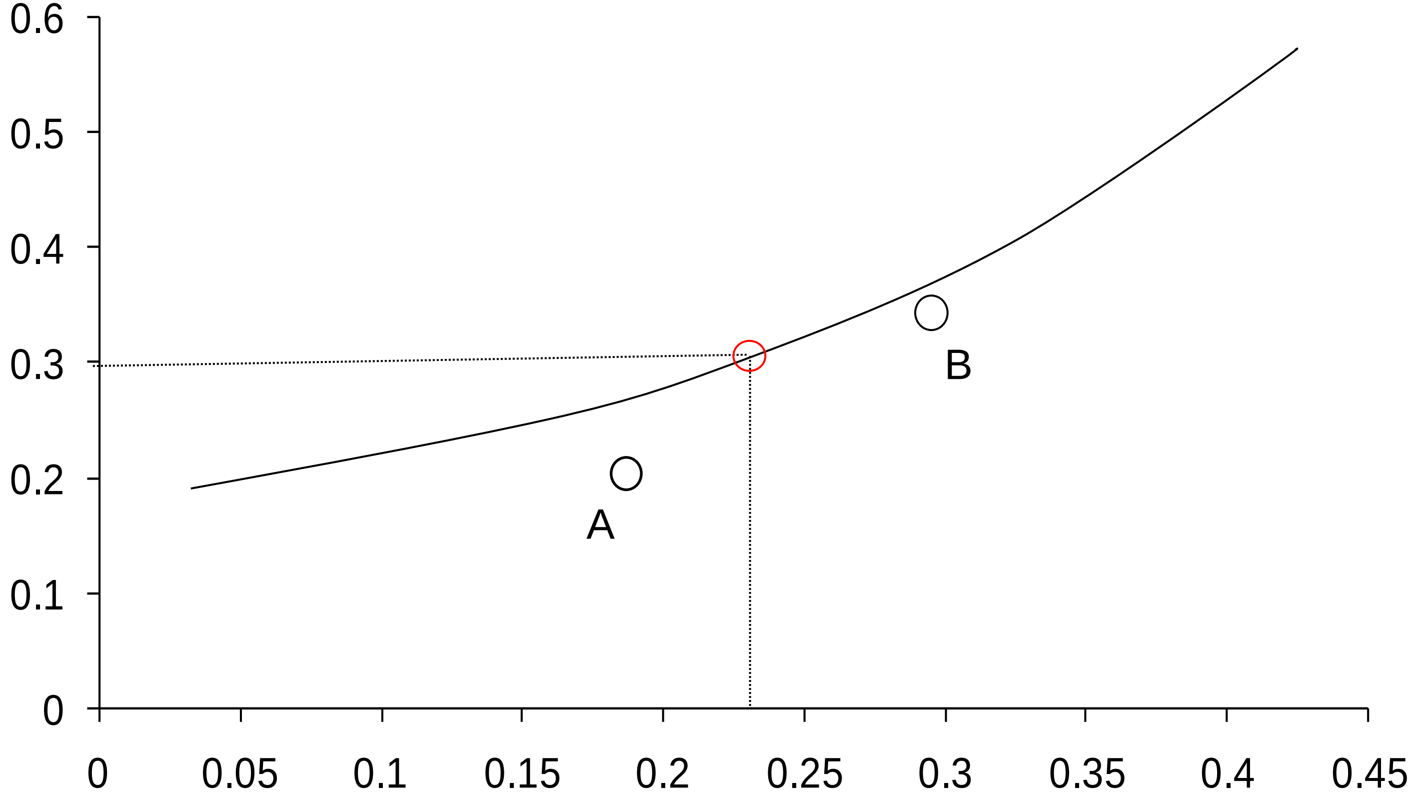
a) Calculate the minimal variance portfolio for a given expected portfolio return of $\bar{\mu}_P = 25\%$

b) What is the variance and the expectation value of this portfolio?

c) Determine the equation of the efficient frontier that can be calculated as the combination of both shares.

d) Which efficient portfolio should an utility-maximizing investor with a preference function of $\Phi(\mu,\sigma) = 1{,}25\mu - 0{,}75(\mu^2 + \sigma^2)$ realize?

Graphical solution for example 1



Portfolio-risks of securities

- For diversified portfolios the risk of an individual security must be assessed and measured in the context of the portfolio as a whole.
- The factor that measures this risk is called portfolio-risk of the stock and is defined as:

$$PR_i = \frac{\mathrm{cov}(r_i, r_p)}{\sigma_P} = \frac{\sigma_i \sigma_P \rho_{iP}}{\sigma_P} = \sigma_i \rho_{iP}$$

- with $-1 \leq \rho_{iP} \leq 1$

- resulting in the following case differentiation:
  - for $\rho_{iP} = 1$ it follows $PR_i = \sigma_i$
  - for $\rho_{iP} < 1$ it follows $PR_i < \sigma_i$
  - for $\rho_{iP} < 0$ it follows $PR_i < 0$, i.e. the security contributes to a risk reduction of the portfolio.

Continuation of example 1

What are the portfolio-risks of securities 1 and 2 for the realized portfolio in part (a)?

Introduction of a risk-free asset to the optimization calculus

- In addition, there is a risk-free asset with a secured return r_f and zero variance
- An investor can obtain funds for this return r_f
- y is the amount of funds invested in the risk-free asset
- The side conditions change as follows:

$$\sum_{i=1}^{N} x_i + y = 1 \qquad \text{and} \qquad \bar{\mu}_P = \sum_{i=1}^{N} x_i \mu_i + y r_f = r_f + \sum_{i=1}^{N} x_i (\mu_i - r_f)$$

- Accordingly, the Lagrange-function to calculate the minimal variance portfolio weights is now defined as:

$$L = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij} + \lambda \left(r_f + \sum_{i=1}^{N} x_i (\mu_i - r_f) - \bar{\mu}_P \right)$$

Tobin Separation Theorem

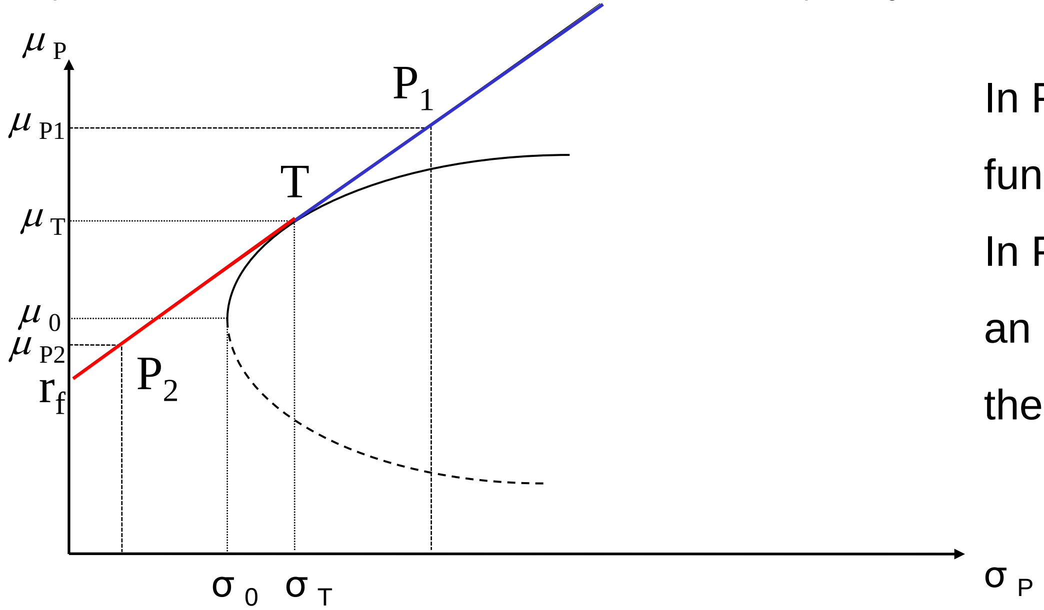
The structure of the optimal stock portfolio is independent from the proportion invested in the risky asset

This means:

- If risk-free investing/borrowing exists alongside risky stock investments, the aspired expected value of return only influences the amount of funds invested in risky stocks.
- The structure of the optimal stockholdings remains unaffected.
- The investor makes his decision about the structure of the optimal stock portfolio independently from the decision about the optimal proportion invested in the stock portfolio.

Tangential portfolio and efficiency line

- Initial question: Which portfolio of risky stocks is realized by the investor?
- This portfolio corresponds to the tangential portfolio which is obtained by constructing the tangent from $\mu_p = r_f$ on the ascending efficient frontier (for $r_f < \mu_0$)



In P_2 the investor invests a part of his funds risk-free at r_f;

In P_1 the investor borrows extra money at an interest rate of r_f in order to invests these funds in risky assets.

- The tangential portfolio is exactly the portfolio of risky assets which is realized
- If the investor wants to achieve μ_{P1} (μ_{P2}) he chooses portfolio P_1 (P_2).
- The structure of P_1 and P_2 is identical to the one of portfolio T, i.e. they are completely positively correlated.

Portfolio-risks of securities

- If a risk-free asset exists, the portfolio-risk of a stock can be calculated in two different ways:

(1) $$PR_i = \frac{\operatorname{cov}(r_i, r_p)}{\sigma_P} = \sigma_i \rho_{iP}$$ (see above)

(2) $$PR_i = \sigma_P \frac{\mu_i - r_f}{\mu_P - r_f}$$

- (2) shows: the stock's portfolio-risk is proportional to its risk premium.

Example 2

In addition to stock 1 and 2 with $E(r_1)=\mu_1=0{,}2$, $E(r_2)=\mu_2=0{,}3$, $var(r_1)=\sigma_1^2=0{,}04$, $var(r_2)=\sigma_2^2=0{,}08$ and $cov(r_1,r_2)=\sigma_{12}=0{,}02$ there is now a capital market providing the opportunity to unboundedly invest and raise capital at a risk-free interest rate of $r_f = 0{,}1$.

a) Calculate the minimal variance portfolio for an expected value of the portfolio return of $\bar{\mu}_P = 25\%$. What is the variance of this portfolio?

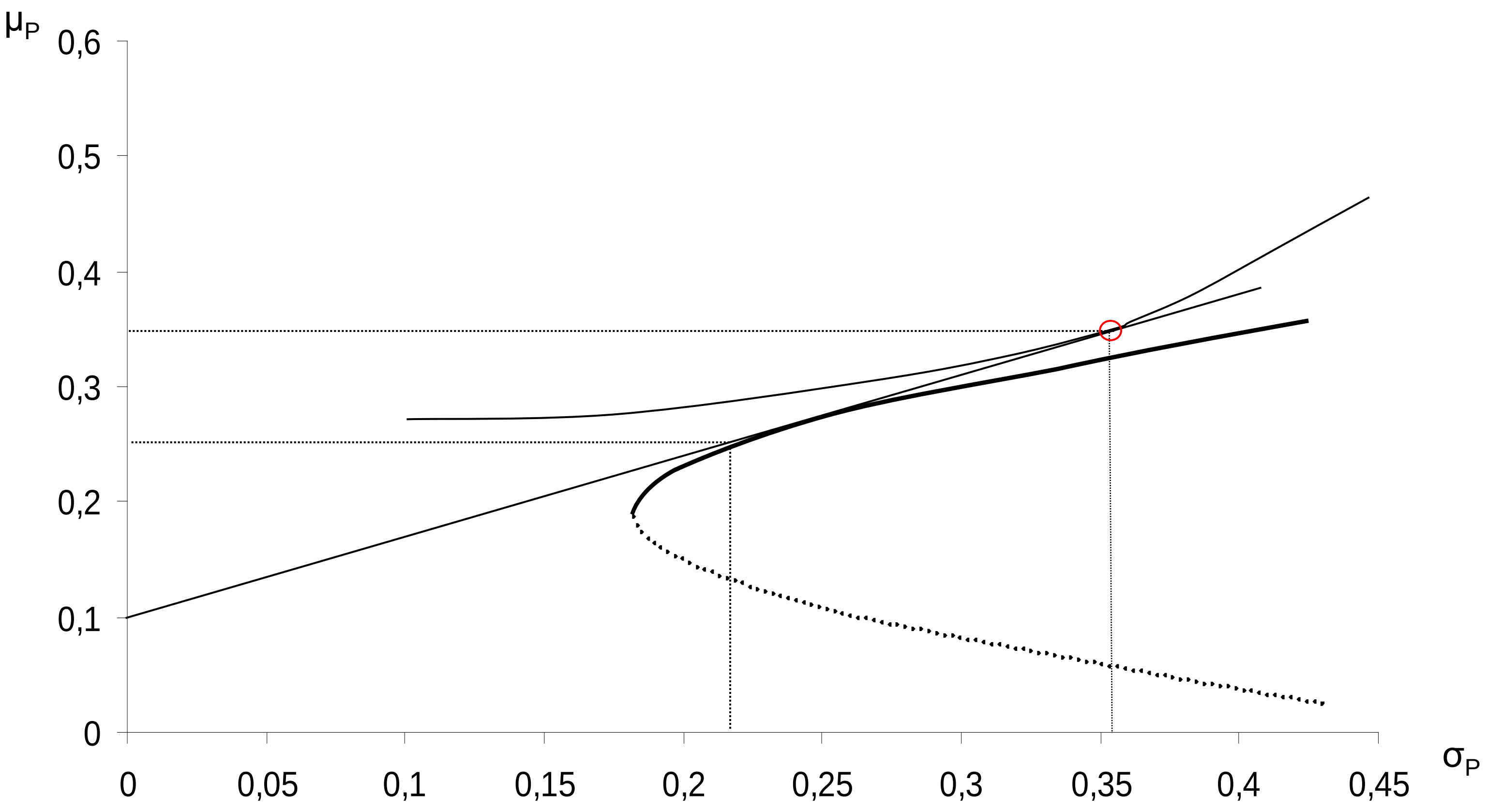
b) Determine the expected value and the variance of the tangential portfolio.

c) Determine the equation for the efficiency line which can be calculated by combining both stocks and the risk-free investment.

d) What are the portfolio-risks of stock 1 and 2 of the portfolio created in a)? How do they correspond to each other?

e) Which of the efficient portfolios should a utility-maximizing investor with a preference function of $\Phi(\mu,\sigma)=1{,}25\mu-0{,}75(\mu^2+\sigma^2)$ realize?

Graphical solution for example 2



- The more stable the input factors of the mean-variance optimization, the easier is the prediction of future values and the more stable are the optimization results.
- An investment strategy based on portfolio selection prefers such stocks within a portfolio the most which
  - have the highest expected returns
  - have the lowest correlation with each other and
  - are characterized by a low variability of return.
- In practice, risk-return-combinations of stocks are subject to estimation errors resulting in suboptimal investment decisions and overestimation of the predicted investment success.
- Application problems of Portfolio Selection Theory have led to the development of alternative approaches (asset allocation).
- Portfolio Selection and asset allocation can be used simultaneously.
- Portfolio Selection for global proportioning does not seem feasible.